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ODE SERIES SOLUTION OF REGULAR SINGULAR POINT

(4) Find the general series solution of the ODE $2x^2y'' - xy' + (1-x^2)y = 0$ about $x=0$

Sol^n:

$$2x^2y'' - xy' + (1-x^2)y = 0 \quad \text{------} \quad (1)$$

Its normalised form is

$$y'' - \frac{x}{2x^2}y' + \frac{1-x^2}{2x^2}y = 0$$

Here, $p(x) = -\dfrac{x}{2x^2}$ (x কাটবে না, অনর্থক থাকবে)

$$Q(x) = \frac{1-x^2}{2x^2}$$

– Both are not analytic at $x=0$

Hence $x=0$ is not an ordinary point.

Now, $(x-0)\cdot P(x) = -\dfrac{x^2}{2x^2} = -\dfrac{1}{2}$

$$(x-0)^2 \cdot Q(x) = \frac{1}{2}(1-x^2)$$

– which are both analytic at $x=0$

Therefore $x=0$ is a regular singular point.

Let us take

$$y = \sum_{n=0}^{\infty} a_n x^{k+n}, \quad a_0 \neq 0 \quad \text{------} \quad (2)$$

be a trial power series solution of (1)

Differentiating ② w.r.t. x, twice we get

$$\cancel{\frac{dy}{dx} = \sum_{n=0}^{\infty} (k+n) a_n x^{k+n-1}}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n-2}$$

putting these in ① we get

$$2x^2 \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n-2}$$
$$- x \sum_{n=0}^{\infty} (k+n) a_n x^{k+n-1} + (1-x^2)\sum_{n=0}^{\infty} a_n x^{k+n} = 0$$

$$\Rightarrow 2 \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n} - \sum_{n=0}^{\infty} (k+n) a_n x^{k+n}$$
$$+ \sum_{n=0}^{\infty} a_n x^{k+n} - \sum_{n=0}^{\infty} a_n x^{k+n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(2(k+n)(k+n-1) a_n - (k+n) a_n + a_n \right) x^{k+n}$$
$$- \sum_{n=0}^{\infty} a_n x^{k+n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[2(k+n)^2 - 3(k+n) + 1 \right] a_n x^{k+n}$$
$$- \sum_{n=0}^{\infty} a_n x^{k+n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (2k+2n-1)(k+n-1) a_n x^{k+n}$$
$$- \sum_{n=0}^{\infty} a_n x^{k+n+2} = 0 \qquad \text{③}$$

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Relation (3) is valid in some deleted neighbourhood of 0.

Thus equating to zero the coefficient of lowest degree term of x i.e x^k in (3) we get

$$(2k-1)(k-1)a_0 = 0 \quad [n \text{ কে } 0 \text{ থেকে start করলে lowest degree term পাবো}]$$

$$\therefore k = \frac{1}{2},\ k = 1 \quad [\because a_0 \neq 0]$$

Now, equating to zero the coefficient of x^{k+n} we get the relation

$$(2k+2n-1)(k+n-1)a_n - a_{n-2} = 0$$

$$\Rightarrow a_n = \frac{a_{n-2}}{(2k+2n-1)(k+n-1)},\ n \geq 2 \quad \text{———} (4)$$

Now, equating to zero the coefficient of x^{k+1} we get the relation

$$(2k+1)(k)a_1 = 0 \Rightarrow a_1 = 0 \quad [\text{for } k = \tfrac{1}{2}, 1] \quad \text{———} (5)$$

using (4) and (5) we get

$$a_1 = a_3 = a_5 = \cdots\cdots = 0$$

Now, putting $n=2$ in (4) we get

$$a_2 = \frac{a_0}{(2k+3)(k+1)} \quad \text{———} (6)$$

putting $n=4$ in (4) we get

$$a_4 = \frac{a_2}{(2k+7)(k+3)} = \frac{a_0}{(2k+3)(k+1)(2k+7)(k+3)}$$

-------------- and so on.

putting these values in ② we get

$$y = a_0 x^k \left[1 + \frac{x^2}{(k+1)(2k+3)} + \frac{x^4}{(2k+3)(k+1)(2k+7)(k+3)} + \cdots\cdots\right]$$

Now, for $k = \frac{1}{2}$, replacing a_0 by c_1 we get

$$y = c_1 x^{1/2}\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} + \cdots\right]$$

$$= c_1 \cdot u$$

for, $k = 1$, replacing a_0 by c_2 we get

$$\Rightarrow \quad y = c_2 x\left[1 + \frac{x^2}{2\cdot 5} + \frac{x^4}{2\cdot 4\cdot 5\cdot 9} + \cdots\cdots\right]$$

$$= c_2 \cdot v$$

$\Rightarrow$ Hence the required complete series solution of the given equation is $y = c_1 u + c_2 v$, where c_1, c_2 are arbitrary constants.